Groups of transformations

Sasha Patotski

Cornell University

ap744@cornell.edu

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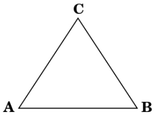
Symmetries of a triangle



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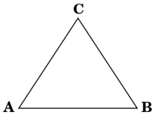

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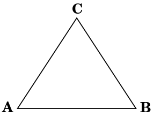

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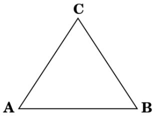

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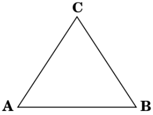

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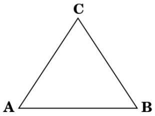

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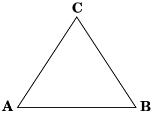

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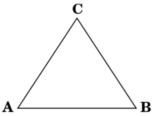

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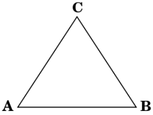

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- Do symmetries s_{AB}, s_{BC} commute?

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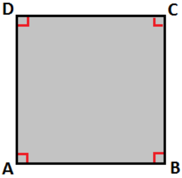

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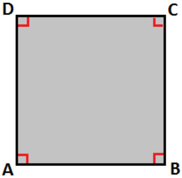

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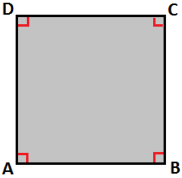

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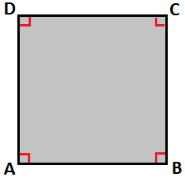

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- What are some symmetries of order 2 and 4?
- Which symmetries reverse orientation of vertices, and which do not?

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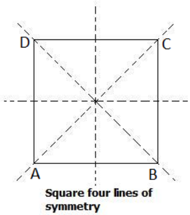

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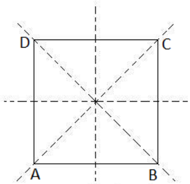

Square four lines of symmetry

- Let c be the symmetry $A \to B \to C \to D \to A$.
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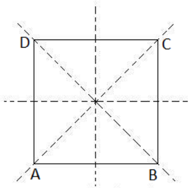

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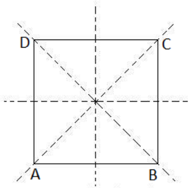

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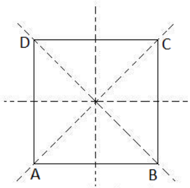

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- **Exercise:** can you express s_h, s_{d1} and s_{d2} using c and s_v?

Flipping mattress

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You want to make a flipping schedule to prevent your magic mattress from becoming a sagging mattress.

Sagging mattress

Let's agree, it looks bad (and probably feels not much better).

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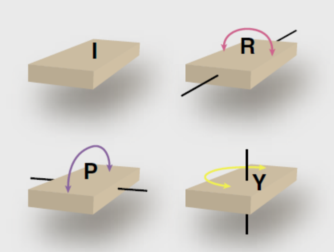

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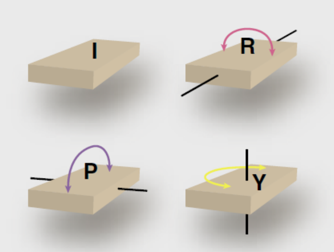

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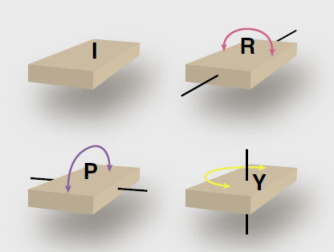

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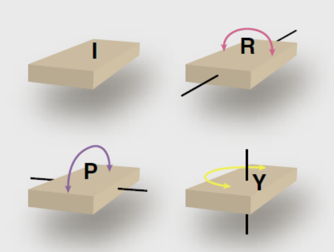

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- Can you get the desired schedule?

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- Symmetries under multiplication form a non-trivial (interesting!) structure.
- Not all symmetries commute.
- Often the set of symmetries (which can be big!) can be expressed in terms of a very few symmetries, which "generate" this set.

Definition of group of transformation

Definition

Let X be a set, and let G be a subset of the set $Bij(X)$ of all bijections $X \to X$. One says G is a **group** if

1. G is closed under composition;
2. $id \in G$;
3. if $g \in G$, then $g^{-1} \in G$.

Example

Symmetries of a triangle, a square and a mattress form a group.

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- Prove that any permutation is a composition of transpositions of neighbors.

Notation

It is convenient to denote permutations by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \dots & \sigma(n) \end{pmatrix}$$

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$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 1 & 6 & 5 & 2 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 6 & 1 & 5 \end{pmatrix}$$

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- Find the inverses of σ_1, σ_2 and $\sigma_2 \circ \sigma_1$.
- Verify that $(\sigma_2 \circ \sigma_1)^{-1} = \sigma_1^{-1} \circ \sigma_2^{-1}$.

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- Prove that for any representation of σ as a composition of N transpositions of neighbors, the sign $sgn(\sigma)$ is $(-1)^N$.
- Prove that for two permutations σ, τ we have $sgn(\sigma \circ \tau) = sgn(\sigma) sgn(\tau)$.